Applying Graph Theory to Tower Defense: A Case Study on Arknights Integrated Strategies

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***Abstract*—Arknights is a mix of tactical RPG and tower defense game in which they have multiple game modes with one of them being Integrated Strategies. Integrated Strategies is a roguelike game mode where players complete a "run" by clearing stages that are divided by branches with different possibilities. This paper provides an approach to analyzing tower defense stage design and dungeon maps using graph theory, showing how graph concepts, such as vertex cuts and directed acyclic graphs, are used in Integrated Strategies Season 2.**

***Keywords*—Arknights, Tower Defense, Stage Design, Graphs, Pathfinding, Vertex Cut, Directed Acyclic Graph, Strategy, Resource Management**

I. Introduction

Arknights is a tactical and tower defense role-playing game (RPG) where the main gameplay is for players to play through multiple chapters of tower defense stages with operators of the player's choice. The uniqueness of Arknights is in its gameplay mechanic in which each stage has its own unique map design and enemies where the player must complete the stage by placing operators on the field that acts as the "tower" in the usual tower defense genre. Each enemy has its own pathfinding towards its goal and the operators that are placed are tasked with preventing the enemies from reaching it.

Fig 1. Integrated Strategies 2: Phantom & Crimson Solitaire. Source: https://arknights.fandom.com/wiki/Phantom_%26_Crimson_Solitaire

In Arknights, there are multiple game modes available to the players other than the main story, one of which is a roguelike mode called Integrated Strategies (IS). Integrated Strategies lets the player make decisions upon the multiple branches of events and encounters while also managing their resources and operators throughout the run. As of the writing of this paper, there have been five playable seasons of IS, excluding the very first season due to it not being made permanent in the game. Specifically, this paper focuses on Integrated Strategies Season 2: Phantom & Crimson Solitaire, as it provides a self-contained game mode encompassing multiple graph structures.

This paper analyzes the mechanics of Integrated Strategies: Phantom & Crimson Solitaire through graph theory, specifically implementing directed acyclic graphs (DAGs) to model the dungeon run map and vertex cuts to formalize enemy pathfinding and optimal operator placement. Through this analysis, it will demonstrate that core gameplay mechanics have precise graph-theoritic structure and provide a formal framework for understanding stage design and progression in tower defense games.

II. Theoretical Framework

A. Graph

In discrete math, a graph is defined as a structure with a set of objects that are related or connected. The objects are referred to as vertices or nodes and the relation or connection are referred to as edges. However, formally, a graph is defined as a G = {V,E} that consists of V, which is a non-empty set of nodes, and E, a set of edges with each having association with two elements of subset V that are called endpoints. Graphs is depicted For example, a graph G = {V, E} where V is {a, b, c, d, e, f} and E is {{a,b}, {a,e}, {a,d}, {b,c}, {c,e}, {c,f}, {e,d}, {e,f}} can be drawn as such in fig 2.1.

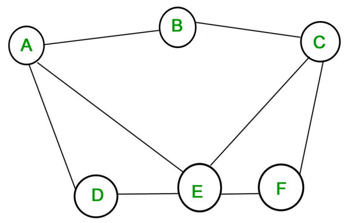

Fig 2.1 Graph G.

Source : https://www.geeksforgeeks.org/engineering-mathematics/mathematics-graph-theory-basics/

B. Graph Types

- Directed graph: A graph with each edge having a direction orientation so that a set of nodes may be connected in one direction but not the other unless it is directed as such.
- Undirected graph: A graph that does not have a direction for each edge so that a set of nodes that have an edge are always connected in all directions.
- Weighted graph: A graph with each edge having a weight or value.
- Unweighted graph: A graph with each edge not having a weight or value.

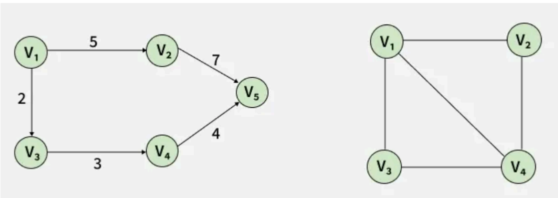

Fig 2.2 (a) Weighted and Directed Graph (b) Unweighted and Undirected Graph.
Source: https://www.geeksforgeeks.org/dsa/graph-data-structure-and-algorithms/

C. Directed Acyclic Graphs (DAGs)

Directed Acyclic Graphs (DAGs) are a type of graph with each edge having a direction that forms an acyclic path. An acyclic path means that no node can reach itself through any path.

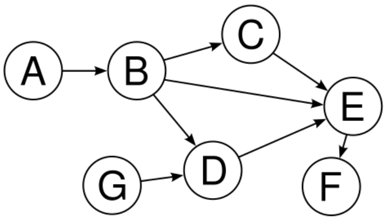

Fig 2.3 Directed Acyclic Graph.

Source:

https://commons.wikimedia.org/wiki/File:Directed_acyclic_graph.svg

D. Subgraph

Subgraph is a graph that is formed from another graph's set of vertices and edges.

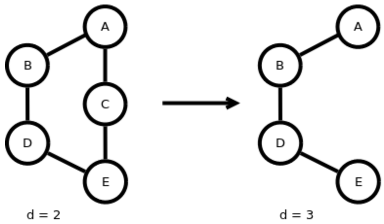

Fig 2.4 (a) Graph G (b) Subgraph of graph G

Source:

https://mathoverflow.net/questions/287022/finding-the-subgraph-with-the-largest-diameter

E. Vertex Cut

Vertex cut is when if and only if a vertex is removed from a graph and it results in two or more disconnected graphs.

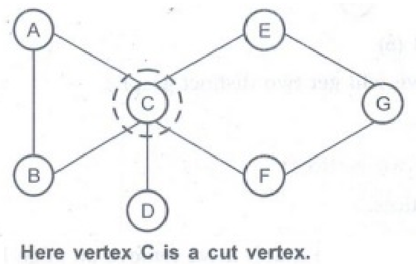

Fig 2.5 Vertex Cut.

Source:

https://cse.poriyaan.in/topic/cut-vertex-50553/

F. Menger's Theorem

In Menger's theorem, the minimum number of nodes where the removal of it will disconnect node S to node T equals the maximum number of node-disjoint paths from S to T.

III. Graph Modelling And Analysis

A. Stage Mechanics and Graph Model

In Arknights, each stage is a two-dimensional tile grid where each traversable tile can be represented as a node in a directed graph G = {V,E}. Edges connect the adjacent traversable tiles in the direction of valid enemy movement that will form a directed graph where enemies travel from spawn point to base as its goal. This movement is modeled as a directed path from source node (spawn) to destination node (base).

Enemy pathfinding is determined by using Shortest Path Faster Algorithm (SPFA), which is a queue-based optimization of the Bellman-Ford algorithm that computes the shortest path from two nodes on a weighted directed graph. Ground enemies will follow its computed path and will reroute if an unpassable object, such as a roadblock, is placed on their path.This confirms that the pathway of enemies is a dynamic weighted directed graph and can be recomputed if a vertex cut happens. This model does not apply to aerial enemies due to there being no mechanic to block its pathway.

The placement of an operator is different from that of a roadblock. An operator has a number of enemies it can block and can be retreated or die, resulting in a dynamic vertex cut. Additionally, if an operator is blocking an enemy's pathway, the enemy does not reroute. For the purpose of this paper, it will be assumed that operator placement has an infinite amount of block counts and can become a vertex cut. It is also assumed that any placements done cannot be removed and creates a static vertex cut. Under these assumptions, finding the optimal placements of operators to block all enemy paths from spawn to base to find the minimum vertex cut required.

B. Integrated Strategies Mechanics and Extended Model

Integrated Strategies: Phantom & Crimson Solitaire introduces additional constraints and mechanics that extends the base graph model. Unlike the main story, the player's squad size is limited and operators must be recruited using Hope, a resource specific to this game mode. These constraints indicate that not all nodes in the minimum vertex cut may be fillable in practice, making this a theoretical minimum vertex cut as a lower bound on the operators needed rather than a guaranteed solution.

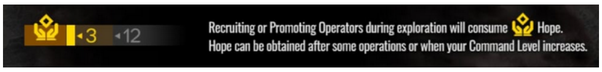

Fig 3.1 Hope System in Phantom & Crimson Solitaire.

Source: Arknights

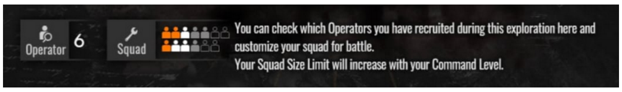

Fig 3.2 Squad System in Phantom & Crimson Solitaire.

Source: Arknights

The Integrated Strategies run itself is structured as a progression map where the player advances through a sequence of encounters and even nodes, such as combat stages, shops, rest, and emergency, before reaching the boss node. Since the progression is always forward and it's impossible to visit previous nodes, the run can be graphed as a Directed Acyclic Graph (DAG) where nodes are represented as the encounters and events, while the progression can be represented as directed edges, therefore there are no existing directed cycles.

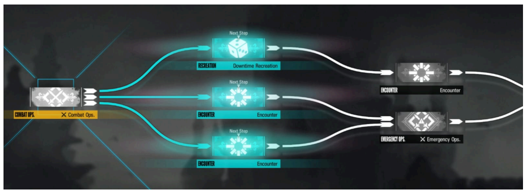

Fig 3.3 Dungeon Run Map in Phantom & Crimson Solitaire

Source: Arknights

One resource that is crucial to manage throughout a run is originium ingots, which is the currency used in the game mode used to purchase items and upgrades in the shop. Certain combat stages within Integrated Strategies contain unowned treasure. If the unowned treasure can be protected throughout the stage, then the player will gain additional originium ingots. The treasure tiles provide additional nodes to the destination nodes where previously the only protected node required is the base.

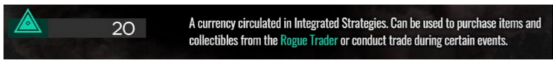

Fig 3.4 Originium Ingots Currency in Phantom & Crimson Solitaire.

Source: Arknights

An Integrated Strategies run will result in a game over if the number of the enemies that reach the base is equivalent to that of the player's available life points. This makes effective stage defense critical to achieve a successful run.

C. Stage Graph

Let the Arknights stage be represented as a finite two-dimensional tile grid of dimension m × n. Each tile will be classified one of the following types: traversable and placeable tile, traversable and unplaceable tile, treasure tile, spawn point, and base. The tiles will be represented as nodes with edges connecting the traversable tiles. There will also be directed edges that indicate the enemies movements. Operator and roadblock placements will be represented as nodes, with roadblock being able to reroute the enemy's pathway based on a weighted graph. Any unplaceable and untraversable tiles are removed from the graph.

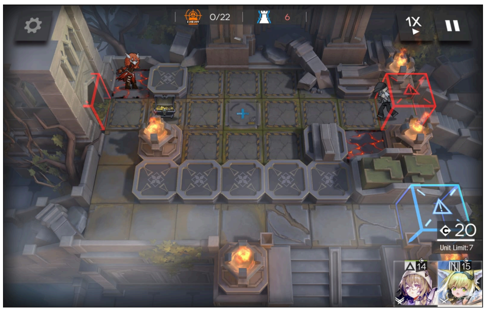

Fig 3.5 “Fight to the Death” Stage
Source: Arknights

For example, in fig 3.7 that is “ Fight to the Death” can be represented as a graph. From the graph representation, it can be determined the minimum cut vertex required to defend the base and the treasure (See fig 3.6).

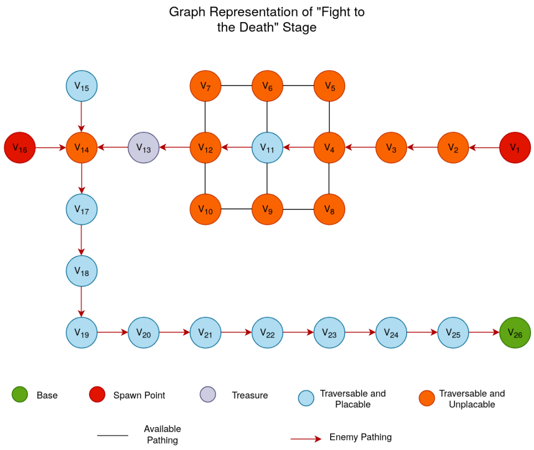

Fig 3.6 Graph Representation of “Fight to the Death” Stage

From the graph representation in fig 3.6, two enemy paths from spawn exist in this stage with one enemy pre-spawned at V_{15}. Path 1 originates from spawn point V_1 and passes directly through treasure tile V_{13} before reaching base V_{26}. Path 2 originates from spawn point V_{16} with a pre-spawned enemy on V_{15}. All paths converge on V_{14} before leading to V_{26}, meaning that the paths are not node-disjoint.

By Menger’s theorem, since the maximum number of node-disjoint paths from source $\{V_1,V_{16}\}$ to base V_{26} is 1, the minimum vertex cut is 1. A single operator placed at any vertex along the shared segment $\{V_{17},...,V_{26}\}$ is sufficient to block all enemy routes to the base.

However, this stage contains an unowned treasure at V_{13} which is directly connected to path 1. To prevent enemies from reaching the treasure, an additional operator must be placed at V_{11}, which is on path 1. This extends the minimum vertex cut to $\{V_{11}, V_{25}\}$ where one operator protects the treasure and one protecting the base.

While Menger’s theorem guarantees a minimum vertex cut of 1 for defending the base alone, if we take path 1 up to the treasure tile and take it as a subgraph first, the minimum cut is 1 on V_{11}. After that, path 1 cannot reach the treasure and the base. Which leaves path 2 and the pre-spawned enemy which path converges in V_{14}, making it a singular path to base. Based on that, it only requires another vertex cut to prevent the remaining enemies from entering the base. In total, with regards to yielding the maximum amount of originium ingots by protecting the treasure, the minimum vertex cut required is 2.

D. Dungeon Run Graph

The Integrated Strategies run map is modeled as a directed graph G = {V,E} where the nodes represent encounter and events while directed edges represent progression between the nodes.

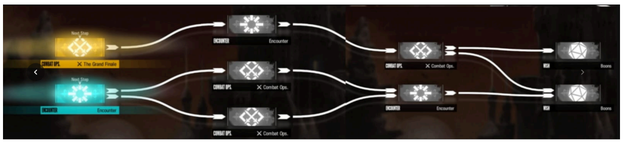

Fig 3.7 “Sunset Courtyard” Dungeon Map

Source: Arknights

Above is the randomized dungeon run that is in Integrated strategies. It consists of 9 nodes across 4 layers in which can be modeled as such (See below fig 3.8)

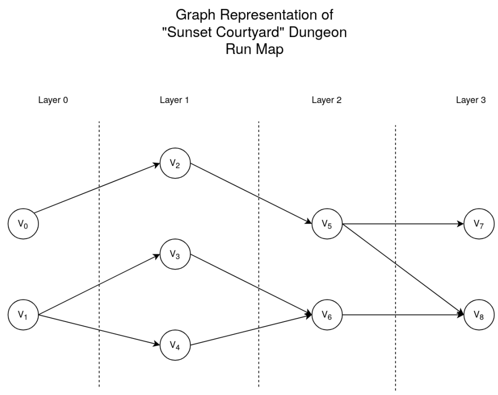

Fig 3.8 Graph Representation of “Sunset Courtyard” Dungeon Run Map

From the graph representation in fig 3.8, it is found that run map progression is a directed acyclic graph as there is no way to backtrack to a previous node. As such it can be found the number of distinct paths the player can take using a dynamic programming approach.

Define that the graph representation in fig 3.8 is G ={V,E} where in layer 0 = $\{V_0,V_1\}$, layer 1 = $\{V_2,V_3,V_4\}$, layer 2 = $\{V_5,V_6\}$, and layer 3 = $\{V_7,V_8\}$. Layer 0 represents the starting choice, while layer 3 represents the end goal. The edges can be defined as E $\{\{V_0,V_2\}, \{V_2,V_5\}, \{V_5,V_7\}, \{V_5,V_8\}, \{V_1,V_3\}, \{V_1,V_4\}, \{V_3V_6\}, \{V_4, V_6\}, \{V_6,V_8\}\}$.

Since G is a DAG, we can apply dynamic programming by setting the value for the starting choice layer's node with 1 and adding it to the next adjacent node. In Layer 0, both nodes start with 1. In layer 1, node V_2 is adjacent only previously to node V_0, so V_2 has a value of 1. This also applies to V_3 and V_4 because they are only previously adjacent to V_1. On layer 2, V_5 has a value of 1 because it is only adjacent previously to V_2. V_6 is previously adjacent to V_3 and V_4 . Therefore V_6 has the added value from V_3 and V_4 , which is 2, because this node can be reached from two different nodes. Finally on layer 3, V_7 has a value of 1 meaning that it can only be reached through a singular path. V_8 can be reached from node V_5 and V_6. Therefore, V_8 has the added value of both nodes, resulting in a value of 3, meaning that V_8 can be accessed through three distinct pathways. In total, there are 4 distinct pathways that can be taken by the player for this particular run map.

From the graph in fig 3.8, it can also be found that there are mandatory encounters the player must face based on the starting choice they pick. If they pick V_0, it will result in two mandatory combats. Otherwise, they will have to engage in a single mandatory combat of their choice between V_3 and V_4. However, there are no mandatory encounters across all four paths. That proves how the starting choice in layer 0 is a consequential decision in the run and determines the mandatory encounters the player must do.

This demonstrates that the DAG structure of Integrated Strategies dungeon run map can provide a basis for strategic route planning. The dynamic programming approach used provides the exact count of available routes with the mandatory nodes that are in relation to the node adjacent to it. This information is useful for squad and resource management, especially on later dungeons and difficulty that would make management harder and the structure of the map more complex.

IV. Conclusion

This paper has demonstrated that graph theory from discrete mathematics can be formally applied to the mechanics of Arknight's Integrated Strategies: Phantom & Crimson Solitaire. Through the construction of a directed graph model from the stage tile grid, the application of Menger's theorem to find the minimum operator placement, and the analysis of the Integrated Strategies dungeon run map as a directed acyclic graph, it has been shown that fundamental mathematical structures underlie the design of a commercial tower defense game, which in the case of Arknights, have been shown.

The model presented in this paper operates under several assumptions, which are infinite operator block count, ground enemies only, and static placement. These bound its applicability to the game's real gameplay scenarios. A dynamic vertex cut model that can count the operator's death or retreat along with their finite block count, as well as a weighted DAG model incorporation node value for resource optimization, would bring the model closer to the actual complexity of the game.

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PERNYATAAN

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Bandung, 19 Juni 2025

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